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Boundary conditions and inversion identities for solvable lattice models with a sublattice symmetry

Je-Young Choi,† Doochul Kim† and Paul A Pearce‡§

† Department of Physics, Seoul National University, Seoul 151-742, Korea
‡ Centre for Mathematical Analysis, Australian National University, Canberra ACT 2601, Australia

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Abstract. The transfer matrices of solvable lattice spin models are considered under general boundary conditions associated with the symmetry of the models. It is shown how the full symmetry can be exploited for systems having a sublattice symmetry by considering an extended model where the states distinguish the sublattices. In this extended scheme, the parity of the number of sites is tied to the boundary conditions. The inversion identities can be extended and be used to obtain the full operater content of the critical models. As an application the complete spectra of the transfer matrices for the eight-vertex model at the decoupling point are calculated.

1. Introduction

There are now several infinite series of solvable classical lattice spin models (Andrews *et al* 1984, Date *et al* 1986, 1987, Pasquier 1987a, b, c, Kuniba and Yajima 1988a, Pearce and Seaton 1988, Jimbo *et al* 1988a, b). In these models, the Boltzmann face weights satisfy the Yang–Baxter equation (YBE) and the row-to-row transfer matrices form a commuting family parametrised by the spectral parameter u (Baxter 1982). At criticality these models realise the conformally invariant quantum field theories in the continuum limit (Cappelli *et al* 1987, Christe and Ravanini 1988). The central charge and the conformal dimensions of the scaling operators of the theory can be determined from eigenvalues of the transfer matrices (Cardy 1986a).

Since this information about the operator content comes from the finite-size corrections, they are sensitive to the boundary conditions (BC). To expose the full operator content of the theory, it is then necessary to consider all the possible BC associated with the symmetry of the model (Cardy 1986b, Rittenberg 1988). When the models possess the sublattice symmetry in the sense defined later, the extra symmetry can be profitably exploited by considering corresponding extended models where the spin states carry an extra index associated with the sublattices. In the next section we discuss the row-to-row transfer matrices under general BC. We observe that the local YBE implies a one-parameter family of commuting transfer matrices for each BC, guaranteeing the solvability of the model for all BC on an equal footing. Then we show how the sublattice symmetry can be handled.

§ Permanent address: Department of Mathematics, University of Melbourne, Parkville, Victoria 3052, Australia.

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The eigenvalues of the transfer matrix may be calculated by using the special functional equations called the inversion identities, satisfied by many solvable models. So far the list of models for which the inversion identities are derived include the Ising model (Baxter 1982), the hard hexagon and the interacting hard square models (Baxter and Pearce 1982, 1983), the eight-vertex model (Pearce 1987a), the self-dual Potts and the Ashkin–Teller models (Pearce 1987b), the magnetic hard square model (Pearce 1985, Pearce and Kim 1987), the restricted SOS models and their fusion hierarchies (Bazhanov and Reshetikhin 1989), and the cyclic SOS models (Pearce and Seaton 1988). (Reshetikhin (1983) considered the inversion identities for the vertex models.) The inversion identities almost completely determine all the zeros of eigenvalues in the complex u plane and hence the complete eigenvalue spectra. Furthermore since these functional equations hold for all finite systems, they can be used to determine the operator content either numerically or analytically. The works mentioned above dealt exclusively with the periodic BC. In this paper (§ 3) we show that the inversion identities can be extended to general cases where the BC associated with the symmetry of each model are imposed.

As an application of the formalism developed in §§ 2 and 3, we consider the eight-vertex model at the decoupling point in § 4. By imposing the general BC, we are able to probe the full dihedral D_4 symmetry of the model. All the eigenvalues are determined exactly for all BC. From this we obtain the full operator context explicitly. The result is then matched to that of the Ashkin–Teller model obtained numerically by Baake *et al* (1987), thereby associating each BC of the eight-vertex model with one of the Ashkin–Teller model.

2. Transfer matrix, symmetry and the extended scheme

Let us consider a square lattice of M rows and N columns wrapped on a torus and associate with each site i a spin variable σ_i that takes on values in the set S of the spin states. The interaction-round-a-face (IRF) model assumes interactions only between spins around a common face (Baxter 1982). The Boltzmann face weight is denoted by $W(a, b, c, d\,|\,u)$, where a, b, c and d are the four spins on a face taken in anticlockwise order starting at the lower left corner, and where u is the spectral parameter that describes the anisotropy of interactions (Kim and Pearce 1987). The symmetry group G of the model is a group of permutations g on the spin-state set S which leave the face weight invariant; thus we have

$$W(ga, gb, gc, gd\,|\,u) = W(a, b, c, d\,|\,u) \tag{2.1}$$

for all $a, b, c, d \in S$ and for all $g \in \mathrm{G}$. This is illustrated in figure 1. (In some models, the symmetry at off-criticality is lower than that at criticality.)

For each $g \in \mathrm{G}$, we define the row-to-row transfer matrix $T_g(u)$ with elements

$$\langle \boldsymbol{\sigma} | T_g(u) | \boldsymbol{\sigma}' \rangle = \prod_{j=1}^{N} W(\sigma_j, \sigma_{j+1}, \sigma'_{j+1}, \sigma'_j | u) \tag{2.2}$$

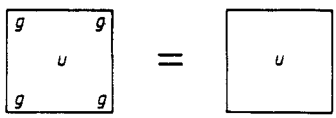

Figure 1. Graphical representation of equation (2.1)

where the row configurations $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N)$ and $\boldsymbol{\sigma}' = (\sigma'_1, \ldots, \sigma'_N)$ satisfy the BC

$$\sigma_{N+1} = g\sigma_1 \qquad \sigma'_{N+1} = g\sigma'_1. \tag{2.3}$$

For many models, there are adjacency conditions which allow only certain pairs of elements in S to occupy adjacent sites. In this case we assume $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ are the allowed configurations. The standard initial condition

$$W(a, b, c, d \mid u = 0) = \delta(a, c) \tag{2.4}$$

yields

$$T_g(u = 0) = C_g \tag{2.5}$$

where the translation (cyclic shift) operator C_g is defined by

$$\langle \boldsymbol{\sigma} | C_g | \boldsymbol{\sigma}' \rangle = \prod_{j=1}^{N} \delta(\sigma_j, \sigma'_{j+1}) \tag{2.6}$$

with $\sigma'_{N+1} = g\sigma'_1$. The BC g lowers the symmetry of the system and the symmetry group of the $T_g(u)$ is given by

$$\mathrm{G}_g = \{g' \in \mathrm{G} \mid gg' = g'g\} \tag{2.7}$$

which is a subgroup of G. Two transfer matrices $T_g(u)$ and $T_{g'}(u)$ have the same spectrum if g and g' belong to the same conjugacy class in G. Not so often emphasised in the literature is that the usual YBE (see figure 2(a)) is sufficient for the commutativity of transfer matrices

$$[T_g(u), T_g(u')] = 0 \tag{2.8}$$

for any BC g as well as for the periodic BC. This follows from the fact that the YBE and the symmetry (2.1) imply

$$\sum_h W(b, gc, gh, a \mid u) W(a, gh, ge, f \mid u') W(h, c, d, e \mid u' - u)$$
$$= \sum_h W(a, b, h, f \mid u' - u) W(b, gc, gd, h \mid u') W(h, gd, ge, f \mid u) \tag{2.9}$$

for all $a, b, c, d, e, f, h \in S$ and for all $g \in \mathrm{G}$. This is illustrated in figure 2. (See de Vega (1984) for a similar idea in vertex models with continuous symmetries.) Hence the solvability applies to all possible BC associated with the symmetry of the model on an equal footing.

For some models, the symmetries discussed above are not sufficient. For example, let us consider the eight-vertex (in IRF version) and the magnetic hard square models. These models are in the same universality class as the Ashkin–Teller model (Kadanoff and Brown 1979, Pearce and Kim 1987, Kim *et al* 1988). The symmetry group of the latter as defined in (2.1) is D_4 whereas it is only Z_2 for the former models. This

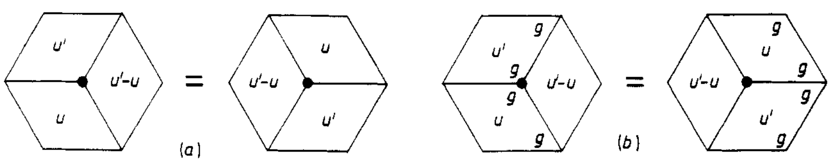

Figure 2. (a) The usual YBE. The spin on the solid circle is summed over. (b) Graphical representation of equation (2.9). Note that figures 1 and 2(a) imply figure 2(b).

discrepancy can be resolved by noting the additional sublattice symmetries in the former models.

Suppose the face weight W has the sublattice symmetry, by which we mean

$$\begin{aligned} W(a,b,c,d\,|\,u) &= W(ga,b,gc,d\,|\,u) \\ &= W(a,gb,c,gd\,|\,u) \end{aligned} \tag{2.10}$$

for all $a, b, c, d \in S$ and for all $g \in \mathrm{G}$. We define a corresponding extended model by assigning an extra 'sublattice' index $\alpha_i = 0, 1$ to each site i. We denote by $\tilde{\sigma}_i = (\sigma_i, \alpha_i)$ the spin state at site i. We then define the face weight of the extended model by

$$\begin{aligned} \tilde{W}(\tilde{\sigma}_i, \tilde{\sigma}_j, \tilde{\sigma}_k, \tilde{\sigma}_l\,|\,u) &= W(\sigma_i, \sigma_j, \sigma_k, \sigma_l\,|\,u)\delta(\alpha_i \oplus 1, \alpha_j)\delta(\alpha_i \oplus 1, \alpha_l) \\ &\quad \times \delta(\alpha_j \oplus 1, \alpha_k)\delta(\alpha_l \oplus 1, \alpha_k) \end{aligned} \tag{2.11}$$

where $\oplus$ represents addition modulo 2. The adjacency condition imposed by the Kronecker delta functions in (2.11) ensures that all sites in one sublattice carry the same sublattice index. If W satisfies the YBE, so does $\tilde{W}$ due to the delta factors. We shall refer to the original (extended) model as the reduced (extended) scheme. Now for the symmetry group elements of $\tilde{W}$, we consider $\tilde{g} = (g_0, g_1, m)$ where $g_0, g_1 \in \mathrm{G}$ and $m \in \mathrm{Z}_2 = \{0, 1\}$. The action of $\tilde{g}$ on a state $\tilde{\sigma} = (\sigma, \alpha)$ is defined by

$$\tilde{g}\tilde{\sigma} = (g_\alpha \sigma, \alpha \oplus m). \tag{2.12}$$

Then the group multiplication law deduced from (2.12) is

$$\tilde{g}\tilde{g}' = (g_{m'} g_0', g_{m'\oplus 1} g_1', m \oplus m') \tag{2.13}$$

where $\tilde{g}' = (g_0', g_1', m')$. The symmetry group $\tilde{\mathrm{G}}$ for $\tilde{W}$ composed of $\tilde{g}$ with such a multiplication law is called the wreath product of G with Z_2 and is denoted by

$$\tilde{\mathrm{G}} = \mathrm{G} \wr \mathrm{Z}_2 \tag{2.14}$$

(Marcu *et al* 1981). For the eight-vertex and the magnetic hard square models with $\mathrm{G} = \mathrm{Z}_2$ we obtain $\tilde{\mathrm{G}} = \mathrm{D}_4$. Hence the full symmetry of these models can be exploited by using the extended scheme.

We next consider the structure of transfer matrices in the extended scheme. Consider a row of N spins $\tilde{\boldsymbol{\sigma}} = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_N)$. It is sufficient to consider two classes of configurations:

$$\begin{aligned} &\text{(i)} \quad \alpha_j = \tfrac{1}{2}(1-(-1)^j) \\ &\text{(ii)} \quad \alpha_j = \tfrac{1}{2}(1+(-1)^j) \end{aligned} \qquad j = 1, \ldots, N. \tag{2.15}$$

The row-to-row transfer matrix $\tilde{T}_{\tilde{g}}(u)$ in the extended scheme is defined as before by

$$\langle \tilde{\boldsymbol{\sigma}} | \tilde{T}_{\tilde{g}}(u) | \tilde{\boldsymbol{\sigma}}' \rangle = \prod_{j=1}^{N} \tilde{W}(\tilde{\sigma}_j, \tilde{\sigma}_{j+1}, \tilde{\sigma}'_{j+1}, \tilde{\sigma}'_j | u) \tag{2.16}$$

with the BC $\tilde{\sigma}_{N+1} = \tilde{g}\tilde{\sigma}_1$ and $\tilde{\sigma}'_{N+1} = \tilde{g}\tilde{\sigma}'_1$. The matrix $\tilde{T}_{\tilde{g}}(u)$ has the block structure

$$\tilde{T}_{\tilde{g}}(u) = \begin{bmatrix} 0 & T_{g_1, g_0}(u) \\ T_{g_0, g_1}(u) & 0 \end{bmatrix} \tag{2.17}$$

where $T_{g_0 g_1}(u)$ are transfer matrices in the reduced scheme defined by

$$\langle \boldsymbol{\sigma} | T_{g_0, g_1}(u) | \boldsymbol{\sigma}' \rangle = \prod_{j=1}^{N} W(\sigma_j, \sigma_{j+1}, \sigma'_{j+1}, \sigma'_j | u) \tag{2.18}$$

with the mixed BC $\sigma_{N+1}=g_0\sigma_1$ and $\sigma'_{N+1}=g_1\sigma'_1$. The YBE implies that the $\tilde{T}_{\tilde{g}}(u)$ form a commuting family. Using (2.17), we then have the quasicommutation relations

$$T_{g_0,g_1}(u)T_{g_1,g_0}(u')=T_{g_0,g_1}(u')T_{g_1,g_0}(u) \tag{2.19}$$

for the transfer matrices in the reduced scheme. We note the consistency condition

$$N=m\ (\mathrm{mod}\ 2). \tag{2.20}$$

Hence the parity of the number N of sites in a row is tied to the boundary conditions. This phenomenon was noted by Kim *et al* (1988), who observed that the operator content of the magnetic hard square model under the periodic BC with odd N is accounted for by that of the Ashkin–Teller model in which the antiperiodic BC is imposed on one of the Ising variables.

The eight-vertex and the magnetic hard square models in the extended scheme can be represented by the adjacency diagrams in figure 3 where each vertex (open circle) corresponds to a spin state and two states can occupy nearest-neighbour sites if they are connected by an edge in the diagrams. Pasquier (1987a, b, c) considered models based on the Dynkin or Coxeter diagrams of simply-laced classical and affine Lie algebras. Among them, A_n, $A^{(1)}_{n-1}(=\hat{A}_{n-1})$ and $D^{(1)}_{n-1}$ $(=\hat{D}_{n-1})$ models with even n can be interpreted as extended schemes for the corresponding $(n/2)$-state models. In fact, the n-state model defined in Akutsu *et al* (1986) and the $2n$-state model in Kuniba and Yajima (1988a) are the reduced and the extended schemes, respectively, of the same model $D^{(1)}_{2n-1}$. The smallest member, the $D^{(1)}_5$ model, corresponds to the magnetic hard square model discussed above. Andrews *et al* (1984) introduced and solved the n-state eight-vertex SOS model A_n $(n=r-1$ in their notation). The lattice-gas representation they proposed for even n is the reduced scheme in our terminology. In particular, the A_4 model corresponds to the interacting hard square model (Baxter and Pearce 1983) and the significance of the parity of N in this model was noted by Kim (1988). The $A^{(1)}_{n-1}$ models correspond to the cyclic SOS models (Pearce and Seaton 1988, Kuniba and Yajima 1988b). Even though the eight-vertex and the $A^{(1)}_3$ models have the same adjacency diagram, their face weights take different forms.

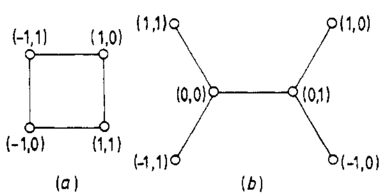

Figure 3. The adjacency diagrams of (a) the eight-vertex model and (b) the magnetic hard square model in the extended scheme. The spin state (σ,α) corresponding to each vertex is shown.

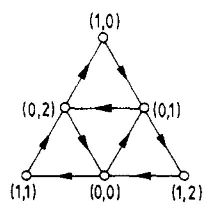

Figure 4. The diagram characterising the spin states and the adjacency condition for the hard hexagon model.

The Z_2 factor in (2.14) can sometimes be generalised to Z_ν. Then the $\oplus$ in (2.11)–(2.13) denotes addition modulo ν with the corresponding change in (2.15), (2.17) and (2.20). A simplest example is the hard hexagon model. Kim (1988) observed that m plays the role of Z_3 BC where $N=m\ (\mathrm{mod}\ 3)$. This is explained by the extended face weight $\tilde{W}$ given in (2.11) where $\oplus$ is addition modulo 3. The corresponding adjacency diagram is a directed graph shown in figure 4 (see Jimbo *et al* 1988a, Pasquier 1988).

3. Inversion identities for the eight-vertex model

In this section, for definiteness, we will describe how the inversion identities of the eight-vertex model under the periodic BC (Pearce 1987a) are generalised for all possible BC. Other models referred to in § 1 (in connection with the inversion identities) have similar extensions. We will denote, for example, by (P5a) the equation (5a) in Pearce (1987a).

Equation (P2) defines the face weight for the eight-vertex model in the reduced scheme. The expression (P3) is replaced by (2.18) where g_0, $g_1 \in G=\{+1,-1\}\simeq Z_2$. We have instead the relations

$$T_{g_0,g_1}(\lambda-u)=T^{\mathrm{T}}_{g_1,g_0}(u) \tag{3.1a}$$

$$T_{g_0,g_1}(u=0)=C_{g_1} \tag{3.1b}$$

$$T_{g_0,g_1}(u=\lambda)=C^{-1}_{g_0} \tag{3.1c}$$

where the transition (cyclic shift) operators C_g are defined in (2.6). Instead of (P4a) we write

$$\langle\boldsymbol{\sigma}|T_{g_0,g_1}(u)T_{g_1,g_0}(u+\lambda)|\boldsymbol{\sigma}'\rangle = \mathrm{Tr}\left(\prod_{j=1}^{N-1} S_{+,+}(\sigma_j,\sigma_{j+1},\sigma'_{j+1},\sigma'_j|u)\right) S_{g_0,g_1}(\sigma_N,\sigma_1,\sigma'_1,\sigma'_N|u) \tag{3.2}$$

where we have, instead of (P4b),

$$\langle\tau_1|S_{g_0,g_1}(\sigma_1,\sigma_2,\sigma'_2,\sigma'_1|u)|\tau_2\rangle = W(\sigma_1,g_0\sigma_2,g_1\tau_2,\tau_1|u)\,W(\tau_1,g_1\tau_2,g_0\sigma'_2,\sigma'_1|u+\lambda). \tag{3.3}$$

Then the equations (P5a) are replaced by

$$\begin{aligned}
S_{g_0,g_1}(\sigma_1,\sigma_2,\sigma_2,\sigma_1|u)&=X(g_0\sigma_2)AX(g_1\sigma_1)\\
S_{g_0,g_1}(\sigma_1,\sigma_2,-\sigma_2,-\sigma_1|u)&=X(g_0\sigma_2)BX(g_1\sigma_1)\\
S_{g_0,g_1}(\sigma_1,\sigma_2,\sigma_2,-\sigma_1|u)&=X(g_0\sigma_2)CX(g_1\sigma_1)\\
S_{g_0,g_1}(\sigma_1,\sigma_2,-\sigma_2,\sigma_1|u)&=X(g_0\sigma_2)DX(g_1\sigma_1).
\end{aligned} \tag{3.4}$$

Then instead of (P6c) we find

$$\langle\boldsymbol{\sigma}|T_{g_0,g_1}(u)T_{g_1,g_0}(u+\lambda)|\boldsymbol{\sigma}\rangle = (c_+c_-s_+s_-)^N+g_0g_1[\mu^{-4}(1-\eta^2)c^2s^2]^N. \tag{3.5}$$

Since all the effects of the BC g_0 and g_1 are contained in the X matrix (see (3.4)), the arguments in Pearce (1987a) are valid for our general cases, yielding the identities

$$T_{g_0,g_1}(u)T_{g_1,g_0}(u+\lambda)=\varphi(\lambda+u)\varphi(\lambda-u)\mathrm{I}+\varphi(u)P_{g_0,g_1}(u) \tag{3.6}$$

where $P_{g_0,g_1}(u)$ are auxiliary matrices whose elements are entire functions of u.

The identities (3.6) contain non-commuting matrices and themselves are not useful unless $g_0=g_1$ (see (2.19)). But in the extended scheme (3.6) are translated into

$$\tilde{T}_{\tilde{g}}(u)\tilde{T}_{\tilde{g}}(u+\lambda)=\varphi(\lambda+u)\varphi(\lambda-u)\tilde{\mathrm{I}}+\varphi(u)\tilde{P}_{\tilde{g}}(u) \tag{3.7}$$

where $\tilde{T}_{\tilde{g}}(u)$ are defined in (2.16) and where

$$\begin{aligned}
\tilde{\mathrm{I}}&=\mathrm{diag}(\mathrm{I},\mathrm{I})\\
\tilde{P}_{\tilde{g}}(u)&=\mathrm{diag}(P_{g_1,g_0}(u),P_{g_0,g_1}(u)).
\end{aligned} \tag{3.8}$$

Then (3.1*b*) and (3.1*c*) become

$$\tilde{T}_{\tilde{g}}(u=0)=\tilde{C}_{\tilde{g}} \qquad \tilde{T}_{\tilde{g}}(u=\lambda)=\tilde{C}_{\tilde{g}}^{-1} \tag{3.9}$$

where

$$\tilde{C}_{\tilde{g}}=\begin{bmatrix} 0 & C_{g_0} \\ C_{g_1} & 0 \end{bmatrix}. \tag{3.10}$$

Since the $\tilde{T}_{\tilde{g}}(u)$ form a commuting family, the inversion identities (3.7) can be used to obtain the eigenvalue spectra for all BC associated with the D_4 symmetry. In the next section, the complete eigenvalue spectra and the operator content are calculated at the decoupling point.

4. Eight-vertex model at the decoupling point

As an application of the formalism developed in § 2, we consider the eight-vertex model at the decoupling point or the doubled Ising model with the face weight

$$W(a,b,c,d\,|\,u)=\exp(-K-L)\exp(Lac+Kbd) \tag{4.1}$$

where $a, b, c, d=\pm 1$. The prefactor $\exp(-K-L)$ is introduced for later convenience. W satisfies (2.10) with $G=\{+1,-1\}\simeq Z_2$. We are interested in the operator content and work in the critical manifold. With the parametrisation

$$\begin{aligned}\exp(2K)&=(1+\sin u)/\cos u\\ \exp(2L)&=(1+\cos u)/\sin u\end{aligned} \tag{4.2}$$

(Baxter 1982), W satisfies the YBE.

Following Baxter (1982), we obtain the inversion identities

$$\begin{aligned}T_{g_0,g_1}(u)T_{g_1,g_0}(u+\pi/2)\\ &=f(u)[(2\cot u)^{N/2}I+g_1(-2\tan u)^{N/2}R_1]\\ &\quad\times[(2\cot u)^{N/2}I+g_0(-2\tan u)^{N/2}R_0]\end{aligned} \tag{4.3}$$

for even N, and

$$T_{g_0,g_1}(u)T_{g_1,g_0}(u+\pi/2)=f(u)[(2\cot u)^N I+g_0g_1(-2\tan u)^N R] \tag{4.4}$$

for odd N. Here and below

$$f(u)=[\tan(u/2)]^N.$$

The spin-reversal operator R and the sublattice-spin-reversal operators R_0 and R_1 are defined by

$$\begin{aligned}\langle\boldsymbol{\sigma}|R|\boldsymbol{\sigma}'\rangle&=\prod_{j=1}^{N}\delta(\sigma_j,-\sigma_j')\\ \langle\boldsymbol{\sigma}|R_0|\boldsymbol{\sigma}'\rangle&=\prod_{j=1}^{N/2}\delta(\sigma_{2j-1},\sigma_{2j-1}')\delta(\sigma_{2j},-\sigma_{2j}')\\ \langle\boldsymbol{\sigma}|R_1|\boldsymbol{\sigma}'\rangle&=\prod_{j=1}^{N/2}\delta(\sigma_{2j-1},-\sigma_{2j-1}')\delta(\sigma_{2j},\sigma_{2j}').\end{aligned} \tag{4.5}$$

The inversion identity (4.4) with $g_0=g_1=+1$ was obtained by Pearce (1983).

In the extended scheme, we work with the transfer matrices $\tilde{T}_{\tilde{g}}$ defined in (2.16) or (2.17) with $\tilde{g}=(g_0, g_1, m)$ and with

$$\tilde{R}_0=\mathrm{diag}(R_0, R_1) \qquad \tilde{R}_1=\mathrm{diag}(R_1, R_0)$$
$$\tilde{R}=\mathrm{diag}(R, R) \qquad \tilde{\mathrm{I}}=\mathrm{diag}(\mathrm{I}, \mathrm{I}). \tag{4.6}$$

The inversion identities (4.3) and (4.4) are translated into

$$\tilde{T}_{\tilde{g}}(u)\tilde{T}_{\tilde{g}}(u+\pi/2) = f(u)[(2\cot u)^{N/2}\tilde{\mathrm{I}}+g_0(-2\tan u)^{N/2}\tilde{R}_0] \times[(2\cot u)^{N/2}\tilde{\mathrm{I}}+g_1(-2\tan u)^{N/2}\tilde{R}_1] \tag{4.7}$$

for $m=0$ ($N=$ even) and

$$\tilde{T}_{\tilde{g}}(u)\tilde{T}_{\tilde{g}}(u+\pi/2)=f(u)[(2\cot u)^{N}\tilde{\mathrm{I}}+g_0g_1(-2\tan u)^{N}\tilde{R}] \tag{4.8}$$

for $m=1$ ($N=$ odd). These identities consist of mutually commuting matrices so they are satisfied for each eigenvalue. Denote by $\tilde{\Lambda}_{\tilde{g}}(u)$, $\tilde{r}$, $\tilde{r}_0$ and $\tilde{r}_1$ the eigenvalues of $\tilde{T}_{\tilde{g}}(u)$, $\tilde{R}$, $\tilde{R}_0$ and $\tilde{R}_1$, respectively.

We first consider even N. Proceeding as in Baxter (1982), we obtain the expression

$$\tilde{\Lambda}_{\tilde{g}}(u)=2F(u)[N^{1/2}\exp(\mathrm{i}\,\pi/4+\mathrm{i}u)]^{\nu_0+\nu_1}\prod_{j=1}^{N-\nu_0-\nu_1}[\exp(2\mathrm{i}u)+\mathrm{i}\gamma_j\tan(\theta_j/2)] \tag{4.9}$$

where

$$\nu_0=\tfrac{1}{2}(1-g_0\tilde{r}_0) \qquad \nu_1=\tfrac{1}{2}(1-g_1\tilde{r}_1)$$
$$\theta_j=(2\pi/N)[j-\tfrac{1}{2}(1-\nu_0)] \qquad j=1,\ldots,(N/2)-\nu_0 \tag{4.10}$$
$$\theta_{N/2-\nu_0+j}=(2\pi/N)[j-\tfrac{1}{2}(1-\nu_1)] \qquad j=1,\ldots,(N/2)-\nu_1.$$

Here $\gamma_j=\pm1$ can be chosen independently only with restrictions

$$\prod_{j=1}^{N/2}\gamma_j=g_0 \qquad \text{if } \nu_0=0$$
$$\prod_{j=1}^{N/2}\gamma_{N/2-\nu_0+j}=g_1 \qquad \text{if } \nu_1=0. \tag{4.11}$$

In (4.9),

$$F(u)=\left[\frac{\exp(\mathrm{i}\pi/2)(\exp(\mathrm{i}u)-1)(\exp(\mathrm{i}u)-\mathrm{i})}{2^{3/2}\exp(2\mathrm{i}u)\sin u\cos u}\right]^N. \tag{4.12}$$

From (4.9) one can extract the central charge $c=1$ and the operator content (Cardy 1986a, Kim and Pearce 1987). We denote by $\chi_{c,h}(q)$ the character for an irreducible representation of the Virasoro algebra $\{L_n\}$ of central charge c with the highest weight h. Then the operator content $Z^{\tilde{r}_0,\tilde{r}_1}_{g_0,g_1}(q)=\mathrm{Tr}\ q^{L_0}\bar{q}^{\bar{L}_0}$ under the boundary condition $\tilde{g}=(g_0, g_1, m=0)$ in the sector with quantum numbers $\tilde{r}_0$ and $\tilde{r}_1$ is found to be

$$Z^{\tilde{r}_0,\tilde{r}_1}_{g_0,g_1}(q)=Z^{\tilde{r}_0}_{g_0}(q)Z^{\tilde{r}_1}_{g_1}(q) \tag{4.13}$$

where $Z^+_g(q)$ and $Z^-_g(q)$ are the operator content of the Ising model under the BC g in the Z_2 even and odd sectors, respectively:

$$Z^+_+(q)=\chi_{1/2,0}(q)\chi_{1/2,0}(\bar{q})+\chi_{1/2,1/2}(q)\chi_{1/2,1/2}(\bar{q})$$
$$Z^-_-(q)=\chi_{1/2,0}(q)\chi_{1/2,1/2}(\bar{q})+\chi_{1/2,1/2}(q)\chi_{1/2,0}(\bar{q}) \tag{4.14}$$
$$Z^-_+(q)=Z^+_-(q)=\chi_{1/2,1/16}(q)\chi_{1/2,1/16}(\bar{q})$$

(Cardy 1986b). From (4.13) it is obvious that the operator content under these four BC ($m=0$) is described by the direct sum of two commuting Virasoro algebras with $c=\frac{1}{2}$, as is expected. Using the result of Baake (1988) we can rewrite (4.13) in terms of the characters $\chi_{1,h}$. Based on a numerical analysis of the Ashkin–Teller quantum chain, Baake *et al* (1987) (see also Rittenberg 1988) obtained the operator content of the various sectors for the eight BC corresponding to the D_4 symmetry. By comparing our result with theirs, we can then associate each BC of the eight-vertex model with one of the Ashkin–Teller model. This is shown in table 1.

Table 1. The correspondence of boundary conditions and sectors between the even-N eight-vertex and the Ashkin–Teller models. See Baake *et al* (1987) for the notations $\Sigma^j C^k$ and $\mathscr{A}$, $\mathscr{B}$, etc.

Boundary conditions		Sectors			
(g_0, g_1)	$\Sigma^j C^k$	$(+,+)$	$(-,-)$	$(+,-)$	$(-,+)$
$(+,+)$	Σ^0	$\mathscr{A}\oplus\mathscr{F}$	$\mathscr{C}\oplus\mathscr{G}$	$\mathscr{H}$	$\mathscr{H}$
$(-,-)$	Σ^2	$\mathscr{C}\oplus\mathscr{G}$	$\mathscr{B}\oplus\mathscr{F}$	$\mathscr{K}$	$\mathscr{K}$
$(+,-)$	$\Sigma^3 C$	$\mathscr{H}$	$\mathscr{K}$	$\mathscr{J}$	$\mathscr{E}$
$(-,+)$	ΣC	$\mathscr{H}$	$\mathscr{K}$	$\mathscr{E}$	$\mathscr{J}$

We now consider odd N. We find that the eigenvalues of the transfer matrix $\tilde{T}_{\tilde{g}}(u)$ are given by

$$\tilde{\Lambda}_{\tilde{g}}(u)=2^{1/2}F(u)[2N\exp(\mathrm{i}\pi/2+2\mathrm{i}u)]^{\nu/2}\prod_{j=1}^{N-\nu}[\exp(2\mathrm{i}u)+\mathrm{i}\gamma_j\tan(\theta_j/2)] \tag{4.15}$$

where

$$\nu=\tfrac{1}{2}(1-g_0g_1\tilde{r})$$
$$\theta_j=(\pi/N)[j-\tfrac{1}{2}(1-\nu)] \qquad j=1,\ldots,N-\nu. \tag{4.16}$$

The $\gamma_j=\pm1$ can be chosen independently with the restriction

$$\prod_{j=1}^{N}\gamma_j=g_0g_1 \qquad \text{if } \nu=0. \tag{4.17}$$

The operator content $Z^{\tilde{r}}_{g_0,g_1}(q)$ under the BC $\tilde{g}=(g_0,g_1,m=1)$ in the sector with quantum number $\tilde{r}$ is given by

$$Z^{\tilde{r}}_{g_0,g_1}(q)=(q\bar{q})^{1/32}Z^{\tilde{r}}_{g_0g_1}(q^{1/2}). \tag{4.18}$$

The operator content under these four BC ($m=1$) is described by the Virasoro algebra $\tilde{L}_n=\frac{1}{2}L_{2n}+\frac{1}{32}\delta(n,0)$ where the L_n generate a Virasoro algebra of $c=\frac{1}{2}$ (Baake 1988). The correspondence with the result of Baake *et al* (1987) (see Rittenberg 1988) is shown in table 2.

Table 2. The correspondence of boundary conditions and sectors between the odd-N eight-vertex and the Ashkin–Teller models. The notation is the same as in table 1.

Boundary conditions		Sectors	
(g_0, g_1)	$\Sigma^j C^k$	+	−
$(+,+)$ $(-,-)$	C $\Sigma^2 C$	$\mathscr{D}\oplus\mathscr{I}$	$\mathscr{H}\oplus\mathscr{K}$
$(+,-)$ $(-,+)$	Σ Σ^3	$\mathscr{H}\oplus\mathscr{K}$	$\mathscr{L}\oplus\tilde{\mathscr{L}}$

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